

ENERGY ASSOCIATED WITH SCHWARZCHILD BLACK HOLE IN A MAGNETIC UNIVERSE*

S. S. Xulu[†]

*Department of Applied Mathematics, University of Zululand,
Private Bag X1001, 3886 Kwa-Dlangezwa, South Africa*

Abstract

In this paper we obtain the energy distribution associated with the Ernst space-time (geometry describing Schwarzschild black hole in Melvin's magnetic universe) in Einstein's prescription. The first term is the rest-mass energy of the Schwarzschild black hole, the second term is the classical value for the energy of the uniform magnetic field and the remaining terms in the expression are due to the general relativistic effect. The presence of the magnetic field is found to increase the energy of the system.

Typeset using REVTeX

* This paper is dedicated to Professor George F. R. Ellis on the occasion of his 60th birthday.

[†]E-mail: ssxulu@pan.uzulu.ac.za

I. INTRODUCTION

Nahmad-Achar and Schutz [1] discussed that the conserved quantities such as the energy, momentum and angular momentum play a very important role as they provide a first integral of the equations of motion. These help to solve difficult problems, for instance, collisions, stability properties of physical systems etc. Obviously, it is desirable to incorporate these quantities in general relativity. However, the study of energy localization or quasi-localization in general relativity has been an intractable problem and is still a subject of active interest. The energy content in a sphere of radius r in a given space-time gives a feeling of the effective gravitational mass that a test particle situated at the same distance from the gravitating object experiences. Recently, Virbhadra [2] discussed the importance of this subject in the context of the *Seifert conjecture* as well as the well-known *hoop conjecture* of Thorne. Attempts aimed at obtaining a meaningful expression for local/quasi-local energy have given rise to different definitions of energy and have resulted in a large number of definitions in the literature (see [3]- [6] and references therein). Canonical energy-momentum, derived from variational formulations of general relativity, leads to non-unique pseudotensor expressions [3]. Einstein's energy-momentum complex, used for calculating the energy distribution in a general relativistic system, was followed by many prescriptions: e.g. Landau and Lifshitz, Papapetrou, Weinberg, and many others (see in [4]). Most of these prescriptions restrict one to make calculations using "Cartesian" coordinates. Coordinate independent definitions of energy have been proposed by Komar [7], Penrose [8], and many others (see in [5]). Bergqvist [9] studied seven different definitions of quasi-local masses for the Reissner-Nordström and Kerr space-times and came to the conclusion that no two of the definitions studied gave the same result. In trying to bring the number of suggested quasilocal energies under control, Hayward [10] listed a number of criteria which should be followed in defining quasilocal masses. Although the subject of energy localization has problems which still remain unresolved, some interesting results have been found in recent years. Virbhadra and his collaborators considered many space-times and have shown that several energy-momentum complexes give the same acceptable result for a given space-time. Virbhadra [11] showed that for the Kerr-Newman metric several definitions yield the same result. Following Virbhadra, Cooperstock and Richardson [3] performed these investigations up to the seventh order of the rotation parameter and reported that several definitions yield the same result. Rosen and Virbhadra [12] and Virbhadra [13] studied the energy distribution in the Einstein-Rosen space-time and got the acceptable result. Similarly, for several other well-known space-times it is known that different energy-momentum complexes give the same result (see [14]- [16] and references therein). Cooperstock [17], Rosen [18], Cooperstock and Israelit [19] initiated the study of the energy of the universe. Rosen [18] obtained the total energy of the closed homogeneous isotropic universe and found that to be zero, which supports the studies by Tryon [20]. Aguirregabiria et al. [4] showed that several energy-momentum complexes give the same result for any Kerr-Schild class metric. The above results are favourable for the importance of energy-momentum complexes. In this paper, we compute the energy distribution in the Ernst space-time.

Melvin's magnetic universe [21] is a solution of the Einstein-Maxwell equations corresponding to a collection of parallel magnetic lines of force held together by mutual gravitation. Thorne [22] studied extensively the physical structure of the magnetic universe and

investigated its dynamical behaviour under arbitrarily large radial perturbations. He showed that no radial perturbation can cause the magnetic field to undergo gravitational collapse to a space-time singularity or electromagnetic explosion to infinite dispersion. Later, Ernst [23] obtained the axially symmetric exact solution to the Einstein-Maxwell equations representing the Schwarzschild black hole immersed in the Melvin's uniform magnetic universe. Virbhadra and Prasanna [24] studied the spin dynamics of charged particles in the Ernst space-time. In this paper we obtain the expression for energy distribution in the Ernst space-time. We use the convention that Latin indices take values from 0 to 3 and Greek indices values from 1 to 3, and take $c = G = 1$.

II. THE EINSTEIN-MAXWELL EQUATIONS AND THE ERNST SOLUTION

The Einstein-Maxwell equations are

$$R_i{}^k - \frac{1}{2} g_i{}^k R = 8\pi T_i{}^k, \quad (1)$$

$$\frac{1}{\sqrt{-g}} \left(\sqrt{-g} F^{ik} \right)_{,k} = 4\pi J^i, \quad (2)$$

$$F_{ij,k} + F_{j,k,i} + F_{k,i,j} = 0, \quad (3)$$

where the energy-momentum tensor of the electromagnetic field is

$$T_i{}^k = \frac{1}{4\pi} \left[-F_{im} F^{km} + \frac{1}{4} g_i{}^k F_{mn} F^{mn} \right]. \quad (4)$$

$R_i{}^k$ is the Ricci tensor and J^i is the electric current density vector.

Ernst [23] obtained an axially symmetric electrovac solution ($J^i = 0$) to these equations describing the Schwarzschild black hole in Melvin's magnetic universe. The space-time is

$$ds^2 = \Lambda^2 \left[\left(1 - \frac{2M}{r} \right) dt^2 - \left(1 - \frac{2M}{r} \right)^{-1} dr^2 - r^2 d\theta^2 \right] - \Lambda^{-2} r^2 \sin^2 \theta d\phi^2 \quad (5)$$

and the Cartan components of the magnetic field are

$$\begin{aligned} H_r &= \Lambda^{-2} B_o \cos \theta, \\ H_\theta &= -\Lambda^{-2} B_o (1 - 2M/r)^{1/2} \sin \theta, \end{aligned} \quad (6)$$

where

$$\Lambda = 1 + \frac{1}{4} B_o^2 r^2 \sin^2 \theta. \quad (7)$$

M and B_o are constants in the solution. The non-zero components of the energy-momentum tensor are

$$\begin{aligned}
T_1^1 &= -T_2^2 = \frac{B_o^2 (2M \sin^2 \theta - 2r \sin^2 \theta + r)}{8\pi\Lambda^4 r}, \\
T_3^3 &= -T_0^0 = \frac{B_o^2 (2M \sin^2 \theta - r)}{8\pi\Lambda^4 r}, \\
T_1^2 &= -T_2^1 = \frac{2B_o^2 (2M - r) \sin \theta \cos \theta}{8\pi\Lambda^4 r}.
\end{aligned} \tag{8}$$

The Ernst solution is a black hole solution ($r = 2M$ is the event horizon). For $B_o = 0$ it gives the Schwarzschild solution and for $M = 0$ it gives the Melvin's magnetic universe. The magnetic field has a constant value B_o everywhere along the axis. Ernst pointed out an interesting feature of this solution. Within the region $2m \ll r \ll B_o^{-1}$, the space is approximately flat and the magnetic field approximately uniform, when $|B_o m| \ll 1$.

To get meaningful results for energy distribution in the prescription of Einstein one is compelled to use "Cartesian" coordinates (see [4] [12], [13], [14] and [18]). It was believed that the results are meaningful only when the space-time studied is asymptotically Minkowskian. However, recent investigations of Rosen and Virbhadra [12], Virbhadra [13], and Aguirregabiria *et al.* [4] showed that many energy-momentum complexes can give the same and appealing results even for asymptotically non-flat space-times. Aguirregabiria *et al.* showed that many energy-momentum complexes give the same results for any Kerr-Schild class metric. There are many known solutions of the Kerr-Schild class which are asymptotically not flat. For example, Schwarzschild metric with cosmological constant. The general energy expression for any Kerr-Schild class metric obtained by them immediately gives $E = M - (\lambda/3)r^3$ where λ is the cosmological constant. This result is very much convincing. $\lambda > 0$ gives repulsive effect whereas $\lambda < 0$ gives attractive effect.

The line element (5) is easily transformed to "Cartesian" coordinates t, x, y, z using the standard transformation

$$\begin{aligned}
r &= \sqrt{x^2 + y^2 + z^2}, \\
\theta &= \cos^{-1} \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right), \\
\phi &= \tan^{-1}(y/x).
\end{aligned} \tag{9}$$

The line element in t, x, y, z coordinates becomes

$$\begin{aligned}
ds^2 &= \Lambda^2 \left(1 - \frac{2M}{r} \right) dt^2 - \left[\Lambda^2 \left(\frac{ax^2}{r^2} \right) + \Lambda^{-2} \left(\frac{y^2}{x^2 + y^2} \right) \right] dx^2 - \left[\Lambda^2 \left(\frac{ay^2}{r^2} \right) + \Lambda^{-2} \left(\frac{x^2}{x^2 + y^2} \right) \right] dy^2 \\
&\quad - \Lambda^2 \left[1 + \frac{2Mz^2}{r^2(r - 2M)} \right] dz^2 - \left[\Lambda^2 \left(\frac{2axy}{r^2} \right) + \Lambda^{-2} \left(-\frac{2xy}{x^2 + y^2} \right) \right] dx dy \\
&\quad - \Lambda^2 \left[\frac{4Mxz}{r^2(r - 2M)} \right] dx dz - \Lambda^2 \left[\frac{4Myz}{r^2(r - 2M)} \right] dy dz,
\end{aligned} \tag{10}$$

where

$$a = \frac{2M}{r - 2M} + \frac{r^2}{x^2 + y^2}. \tag{11}$$

III. THE EINSTEIN ENERGY-MOMENTUM COMPLEX

Einstein obtained an energy-momentum complex Θ_i^k , which satisfies the local conservation laws (see in [25])

$$\frac{\partial \Theta_i^k}{\partial x^k} = 0, \quad (12)$$

where

$$\Theta_i^k = \sqrt{-g} (T_i^k + \tau_i^k) \quad (13)$$

Θ_i^k is referred to as the Einstein energy-momentum complex. τ_i^k is usually called *energy-momentum pseudotensor*. T_i^k is the energy-momentum tensor appearing in the Einstein's field equations.

Einstein found that

$$\Theta_i^k = \frac{1}{16\pi} Z_{i, \quad l}^{kl} \quad (14)$$

where

$$Z_i^{kl} = -Z_i^{lk} = \frac{g_{in}}{\sqrt{-g}} \left[-g \left(g^{kn} g^{lm} - g^{ln} g^{km} \right) \right]_{,m} \quad (15)$$

The energy E and the three components of momenta P_α are given by the expression

$$P_i = \int \int \int \Theta_i^0 dx^1 dx^2 dx^3. \quad (16)$$

thus the energy E for a stationary metric is given by the expression

$$E = \frac{1}{16\pi} \int \int \int Z_{0,\alpha}^{0\alpha} dx dy dz. \quad (17)$$

and after applying the Gauss theorem, one has

$$E = \frac{1}{16\pi} \int \int Z_0^{0\alpha} \mu_\alpha dS. \quad (18)$$

For $r = \text{constant}$ surface (given by (9)) one has the components of a normal vector $\mu_\alpha = (x/r, y/r, z/r)$. The infinitesimal surface element is $dS = r^2 \sin\theta d\theta d\phi$.

IV. CALCULATIONS

We have already discussed that to use the energy-momentum complex of Einstein one is compelled to use ‘‘Cartesian’’ coordinates and therefore we consider the Ernst metric in t, x, y, z coordinates, expressed by the line element (10). The determinant of the metric tensor is given by

$$g = -\Lambda^4 \quad (19)$$

The non-zero contravariant components of the metric tensor are

$$\begin{aligned}
g^{00} &= \Lambda^{-2} \frac{r}{r-2M}, \\
g^{11} &= \Lambda^{-2} \left[\frac{2Mx^2}{r^3} - \frac{x^2}{x^2+y^2} \right] - \Lambda^2 \left[\frac{y^2}{x^2+y^2} \right], \\
g^{12} &= \Lambda^{-2} \left[\frac{2Mxy}{r^3} - \frac{xy}{x^2+y^2} \right] + \Lambda^2 \left[\frac{xy}{x^2+y^2} \right], \\
g^{22} &= -\Lambda^{-2} \left[\frac{2My^2}{r^3} - \frac{y^2}{x^2+y^2} \right] - \Lambda^2 \left[\frac{x^2}{x^2+y^2} \right], \\
g^{33} &= \Lambda^{-2} \left[\frac{2Mz^2}{r^3} - 1 \right], \\
g^{13} &= \Lambda^{-2} \left[\frac{2Mxz}{r^3} \right], \\
g^{23} &= \Lambda^{-2} \left[\frac{2Myz}{r^3} \right].
\end{aligned} \tag{20}$$

The only required components of Z_k^{ij} in the calculation of energy are the following:

$$\begin{aligned}
Z_0^{01} &= \frac{4Mx}{r^3} + (\Lambda^4 - 1) \left[\frac{x}{x^2+y^2} \right], \\
Z_0^{02} &= \frac{4My}{r^3} + (\Lambda^4 - 1) \left[\frac{y}{x^2+y^2} \right], \\
Z_0^{03} &= \frac{4Mz}{r^3}.
\end{aligned} \tag{21}$$

Now using (21) with (18) we obtain the energy distribution in the Ernst space-time.

$$E = M + \frac{1}{16\pi} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} (\Lambda^4 - 1) r \sin \theta d\theta d\phi \tag{22}$$

We substitute the value of Λ in the above and then integrate. We get

$$E = M + \frac{1}{6} B_o^2 r^3 + \frac{1}{20} B_o^4 r^5 + \frac{1}{140} B_o^6 r^7 + \frac{1}{2520} B_o^8 r^9. \tag{23}$$

The above result is expressed in geometrized units (gravitational constant $G = 1$ and the speed of light in vacuum $c = 1$). In the following we restore G and c and get

$$E = Mc^2 + \frac{1}{6} B_o^2 r^3 + \frac{1}{20} \frac{G}{c^4} B_o^4 r^5 + \frac{1}{140} \frac{G^2}{c^8} B_o^6 r^7 + \frac{1}{2520} \frac{G^3}{c^{12}} B_o^8 r^9. \tag{24}$$

The first term Mc^2 is the rest-mass energy of the Schwarzschild black hole, the second term $\frac{1}{6} B_o^2 r^3$ is the well-known classical value of the energy of the magnetic field under consideration, and rest of the terms are general relativistic corrections. For very large $B_o r$, the general relativistic contribution dominates over the classical value for the magnetic field energy. As mentioned in Section 2, the gravitational field is weak for $2m \ll r \ll B_o^{-1}$ (in $G = 1, c = 1$ units). Thus in the weak gravitational field we have $B_o r \ll 1$; therefore, the classical value for the magnetic field energy will be greater than the general relativistic correction in these cases.

V. DISCUSSION AND SUMMARY

The energy-momentum localization subject has been associated with much debate. Misner et al. [26] argued that the energy is localizable only for spherical systems. Cooperstock and Sarracino [27] contradicted their viewpoint and argued that if the energy is localizable in spherical systems then it is also localizable for all systems. Bondi [28] noted that a nonlocalizable form of energy is not admissible in relativity. Therefore its location can be found. Several quasi-local mass definitions were proposed (notably by Penrose and Hawking). However, these have some problems (see [9] and [2]). The viewpoints of Misner et al. discouraged further study of the energy localization problem. The energy-momentum complexes are not tensorial objects and one is compelled to use “Cartesian” coordinates. Due to these reasons, this subject remained in an almost “slumbering” state for a long period of time and was re-opened by remarkable results obtained by Virbhadra and Virbhadra and his collaborators (Rosen, Parikh, Chamorro and Aguirregabiria). These works motivated many to come back to this very interesting and important subject (for instance, see [3], [6], [15]- [18], [29], [30] and references given there). Although the energy-momentum complexes are not tensorial objects, they obey conservation laws (for example, see Eq. (12)) which are true in all coordinate systems.

In the present paper we considered the Ernst space-time and calculated the energy distribution using the Einstein energy-momentum complex. It beautifully yields the expected result: The first term is the Schwarzschild rest-mass energy, the second term is the classical value for energy due to the uniform magnetic field ($E = \frac{1}{8\pi} \int \int \int B_o^2 dV$, where dV is the infinitesimal volume element, yields exactly the same value as the second term of (24)), and the rest of the terms are general relativistic corrections. The general relativistic terms increase the value of the energy. Thus, the result in this paper is against the prevailing “folklore” that the energy-momentum complexes are not useful to obtain meaningful energy distribution in a given geometry.

ACKNOWLEDGMENTS

I am grateful to K. S. Virbhadra for guidance, George F. R. Ellis for hospitality at the university of Cape Town, and NRF for financial support.

REFERENCES

- [1] E. Nahmad-Achar and B. F. Schutz, *Gen. Relativ. Gravit.* **19**, 655 (1987).
- [2] K. S. Virbhadra, *Phys. Rev.* **D60**, 104041 (1999); gr-qc/9809077.
- [3] F. I. Cooperstock and S. A. Richardson, in *Proc. 4th Canadian Conf. on General Relativity and Relativistic Astrophysics* (World Scientific, Singapore, 1991).
- [4] J. M. Aguirregabiria, A. Chamorro, and K. S. Virbhadra, *Gen. Relativ. Gravit.* **28**, 1393 (1996).
- [5] J. D. Brown and J. W. York, Jr., *Phys. Rev.* **D47**, 1407 (1993).
- [6] C. Chen and J. M. Nester, gr-qc/9809020.
- [7] A. Komar, *Phys. Rev.* **113**, 934 (1959).
- [8] R. Penrose, *Proc. Roy. Soc. London* **A381**, 53 (1982).
- [9] G. Bergqvist, *Class. Quantum Gravit.* **9**, 1753 (1992).
- [10] S. A. Hayward, *Phys. Rev.* **D49**, 831 (1994).
- [11] K. S. Virbhadra, *Phys. Rev.* **D42**, 1066 (1990); *Phys. Rev.* **D42**, 2919 (1990).
- [12] N. Rosen and K. S. Virbhadra, *Gen. Relativ. Gravit.* **25**, 429 (1993).
- [13] K. S. Virbhadra, *Pramana - J. Phys.* **45**, 215 (1995).
- [14] K. S. Virbhadra and J. C. Parikh, *Phys. Lett.* **B317**, 312 (1993); *Phys. Lett.* **B331**, 302 (1994); K. S. Virbhadra, *Phys. Lett.* **A157**, 195 (1991); *Mathematics Today* **9**, 39 (1991); F. I. Cooperstock, in *Topics on Quantum Gravity and Beyond*, Essays in honour of L. Witten on his retirement, edited by F. Mansouri and J. J. Scanio (World Scientific, Singapore, 1993); K. S. Virbhadra, *Pramana-J. Phys.* **38**, 31 (1992); A. Chamorro and K. S. Virbhadra, *Pramana-J. Phys.* **45**, 181 (1995); A. Chamorro and K. S. Virbhadra, *Int. J. Mod. Phys.* **D5**, 251 (1996); K. S. Virbhadra, *Int. J. Mod. Phys.* **A12**, 4831 (1997).
- [15] S. S. Xulu, *Int. J. Theor. Phys.* **37**, 1773 (1998).
- [16] S. S. Xulu, *Int. J. Mod. Phys.* **D7**, 773 (1998).
- [17] F. I. Cooperstock, *Gen. Rel. Gravit.*, **26**, 323 (1994).
- [18] N. Rosen, *Gen. Relativ. Gravit.* **26**, 319 (1994).
- [19] F. I. Cooperstock and M. Israelit, *Found. of Phys.* **25**, 631 (1995).
- [20] E. P. Tryon, *Nature* **246**, 396 (1973).
- [21] M. A. Melvin, *Phys. Lett.* **8**, 65 (1964).
- [22] K. S. Thorne, *Phys. Rev.* **139**, B244 (1965).
- [23] F. J. Ernst, *J. Math. Phys.* **15**, 1409 (1974); **17**, 54 (1976).
- [24] K. S. Virbhadra and A. R. Prasanna, *Pramana-J. Phys.* **33**, 449 (1989).
- [25] C. Møller, *Annals of Physics* (NY) **4**, 347 (1958).
- [26] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (W. H. Freeman and Co., NY, 1973) p.603.
- [27] F. I. Cooperstock and R. S. Sarracino, *J. Phys.* **A11**, 877 (1978).
- [28] H. Bondi, *Proc. Roy. Soc. London* **A427**, 249 (1990).
- [29] N. Banerjee and S. Sen, *Pramana- J. Phys.* **49**, 609 (1997).
- [30] I. C. Yang, C. T. Yeh, R. R. Hsu and C. R. Lee, gr-qc/9609038, *Int. J. Mod. Phys.* **D**, to appear; I. C. Yang, W-F Lin and R. R. Hsu, gr-qc/9707044.